TRACE-CLASS APPROACH IN SCATTERING PROBLEMS FOR PERTURBATIONS OF MEDIA

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Abstract. We consider the operators $H_0 = M_0^{-1}(x)P(D)$ and $H = M^{-1}(x)$ $P(D)$ where $M_0(x)$ and $M(x)$ are positively definite bounded matrix-valued functions and $P(D)$ is an elliptic differential operator. Our main result is that the wave operators for the pair H_0, H exist and are complete if the difference $M(x) - M_0(x) = O(|x|^{-\rho})$, $\rho > d$, as $|x| \to \infty$. Our point is that no special assumptions on $M_0(x)$ are required. Similar results are obtained in scattering theory for the wave equation.

1. Introduction

There are two essentially different methods in scattering theory: the smooth and the trace-class (see [14], for a more thorough discussion). The first of them originated in the Friedrichs-Faddeev model where a perturbation of the operator of multiplication H_0 by an integral operator V with smooth kernel is considered. The second goes back to the fundamental Kato-Rosenblum theorem which states that the wave operators for the pair H_0, $H = H_0 + V$ exist for a perturbation V from the trace class. In applications to differential operators the smooth method works if the operator H_0 has constant coefficients and coefficients of V tend to zero sufficiently rapidly at infinity. The trace method does not require that coefficients of the operator H_0 be constant, but its assumptions on the fall-off of coefficients of V at infinity are more stringent. The advantages of the trace-class method were discussed in [4] for the case where $H_0 = -\Delta + v_0(x)$ is the Schrödinger operator with an arbitrary bounded potential $v_0(x)$, $x \in \mathbb{R}^d$, and V is a first-order differential operator with coefficients bounded by $|x|^{-\rho}$, $\rho > d$, as $|x| \to \infty$.

Our goal here is to apply the trace-class theory to scattering of waves (electromagnetic, acoustic, etc.) in inhomogeneous media. To be more precise, we consider in §4 the operators

$$H_0 = M_0^{-1}(x)P(D) \quad \text{and} \quad H = M^{-1}(x)P(D). \tag{1.1}$$

Here $M_0(x)$ and $M(x)$ are positively definite bounded matrix-valued functions and $P(D)$ is an elliptic differential operator. The operators H_0 and H are self-adjoint in Hilbert spaces with scalar products defined naturally in terms of M_0 and M, respectively. Our main result is that the wave operators for the pair H_0, H exist, are isometric and complete if

$$|M(x) - M_0(x)| \le C(1 + |x|)^{-\rho}, \quad \rho > d, \quad x \in \mathbb{R}^d, \tag{1.2}$$

(C is some constant). We emphasize that no special assumptions on $M_0(x)$ are required, that is the "background" medium might be inhomogeneous. In contrast, the smooth theory relies on a sufficiently explicit diagonalization of the operator H_0; for example, if M_0 does not depend on x, then H_0 can be diagonalized by the Fourier transform. On the other hand, in this approach it suffices (see [6]) to suppose that $\rho > 1$ in the estimate (1.2).

In §5 we study the scattering theory for the wave equation. This problem can be almost reduced to that considered in §4. However the corresponding operator $P(D)$ is not a differential operator and, if considered as a pseudo-differential operator, it has a non-smooth symbol. It creates some new difficulties.

Both these problems were considered by M. Sh. Birman in the papers [2, 3] where it was supposed that $M_0(x)$ is a constant matrix (there was also a similar assumption in the case of the wave equation). Essentially the same assumptions were made in the papers [12, 10, 7] (see also the book [11]). Similarly to [2, 3], we proceed from the conditions for the existence and completeness of the wave operators established earlier in the paper [1] by A. L. Belopolskii and M. Sh. Birman. However a verification of these conditions for the pair (1.1) in the case where M_0 is a function of x is a relatively tricky business. One of possible tricks is presented in this paper.

Typically, one has to check that the operators $\langle x\rangle^{-r}(H_0 - z)^{-n}$ and $\langle x\rangle^{-r}(H - z)^{-n}$ belong to the Hilbert-Schmidt class $\mathfrak{S}_2$ if $r > d/2$ and n is sufficiently large. Since properties of the operators H_0 and H are the same, we discuss only the second of these operators. The problem here is that the inclusion $\langle x\rangle^{-r}(P(D) - z)^{-n} \in \mathfrak{S}_p$ ($\mathfrak{S}_p$ is the Neumann-Schatten class) implies that $\langle x\rangle^{-r}(H - z)^{-n} \in \mathfrak{S}_p$ for $n = 1$ only. If $n > 1$, then a direct verification of this assertion requires restrictive assumptions on derivatives of $M(x)$. Roughly speaking, we fix this difficulty in the following way. We consider the whole scale of classes $\mathfrak{S}_p$ and find such a number $p(r, n)$ that $\langle x\rangle^{-r}(H - z)^{-n} \in \mathfrak{S}_p$ for $p > p(r, n)$. This is done successively for $n = 1, 2, \dots$. To make a passage from n to $n + 1$, we use that the operators

$$(P(D) - z)\langle x\rangle^{-r}(P(D) - z)^{-1}\langle x\rangle^{r}$$

are bounded for all $r \geq 0$. This allows us to deduce that $\langle x\rangle^{-r}(H - z)^{-n-1} \in \mathfrak{S}_p$ for $p > p(r, n)$ from the inclusions $\langle x\rangle^{-r_0}(H - z)^{-1} \in \mathfrak{S}_p$ for $p > p(r_0, 1)$ and $\langle x\rangle^{-r_1}(H - z)^{-n} \in \mathfrak{S}_p$ for $p > p(r_1, n)$ with suitably chosen $r_0 + r_1 = r$.

2. Preliminaries

1. Let $\mathcal{H}_0$ and $\mathcal{H}$ be two Hilbert spaces, and let $\mathfrak{B}$ be the algebra of all bounded operators acting from $\mathcal{H}_0$ to $\mathcal{H}$. The ideal of compact operators will be denoted by $\mathfrak{S}_\infty$. For any compact A we denote by $s_n(A)$ the eigenvalues of the positive compact operator $(A^*A)^{1/2}$ listed with account of multiplicity in decreasing order. Important symmetrically normed ideals $\mathfrak{S}_p$, $1 \leq p < \infty$, of the algebra $\mathfrak{B}$ are formed by operators $A \in \mathfrak{S}_\infty$ for which

$$\sum_{n=1}^{\infty} s_n^p(A) < \infty.$$

In particular, $\mathfrak{S}_1$ and $\mathfrak{S}_2$ and called the trace and Hilbert-Schmidt classes, respectively. Clearly, $\mathfrak{S}_{p_1} \subset \mathfrak{S}_{p_2}$ for $p_1 \leq p_2$. Moreover, we have

Proposition 2.1. *If $A_j \in \mathfrak{S}_{p_j}$, $j = 1, 2$, and $p^{-1} = p_1^{-1} + p_2^{-1} \leq 1$, then $A = A_1 A_2 \in \mathfrak{S}_p$.*

2. We need to consider integral operators of the form

$$(Tf)(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} a(x) \exp(i\langle x, \xi\rangle) b(\xi) \hat{f}(\xi) d\xi \tag{2.1}$$

acting in the space $L_2(\mathbb{R}^d; \mathbb{C}^k)$. Here

$$\hat{f}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \exp(-i\langle x, \xi\rangle) f(x) dx$$

is the Fourier transform of $f \in L_2(\mathbb{R}^d; \mathbb{C}^k)$ and $a(x)$, $b(\xi)$ are $k \times k$-matrix-functions which we always suppose to be bounded. Then operator (2.1) is bounded. Below we sometimes use the short-hand notation $T = a(x)b(\xi)$ for operators of the form (2.1). Let us also set

$$\langle x\rangle = (1 + |x|^2)^{1/2}, \quad \langle \xi\rangle = (1 + |\xi|^2)^{1/2}.$$

The following assertion is well-known.

Proposition 2.2. *The operator* (2.1) *is compact if the functions a and b tend to zero at infinity. This operator belongs to the class $\mathfrak{S}_p(L_2(\mathbb{R}^d; \mathbb{C}^k))$, $p \geq 1$, if*

$$|a(x)| \leq C(1 + |x|)^{-r}, \quad |b(\xi)| \leq C(1 + |\xi|)^{-r}, \quad r > d/p.$$

The proof of this result can be found, e.g., in [11]. Strictly speaking, the case $p \in (1, 2)$ was not considered in [11]. However it can be directly deduced from Proposition 2.2 for $p = 1$ and $p = 2$ by the complex interpolation.

3. We need also conditions of boundedness in the space $L_2(\mathbb{R}^d; \mathbb{C}^k)$ of products of multiplication operators in x- and ξ-representations.

Proposition 2.3. *Suppose that a matrix-function $a(x)$ has n bounded derivatives and $0 \leq l \leq n$. Then the product $\langle\xi\rangle^l a(x) \langle\xi\rangle^{-l}$ defined by its sesquilinear form on the Schwartz class $\mathcal{S}$ is a bounded operator. Its norm is estimated by*

$$\sup_{|\sigma| \leq n} \sup_{x \in \mathbb{R}^d} |(\partial^\sigma a)(x)|.$$

Proof. Note first that, for all $j = 1, \dots, d$,

$$D_j^n a(x) \langle\xi\rangle^{-n} = \sum_{m=0}^{n} i^{-m} C_n^m \partial^m a(x)/\partial x_j^m (\xi_j^{n-m} \langle\xi\rangle^{-n}), \tag{2.2}$$

where C_n^m are binomial coefficients. Since the functions $\partial^m a(x)/\partial x_j^m$ and $\xi_j^{n-m}\langle\xi\rangle^{-n}$ are bounded, operator (2.2) is also bounded. This entails that the operators $|\xi_j|^n a(x) \langle\xi\rangle^{-n}$ and hence $\langle\xi\rangle^n a(x) \langle\xi\rangle^{-n}$ are bounded.

To pass to an arbitrary l, we consider the function

$$(a(x)\langle\xi\rangle^{-z} f, \langle\xi\rangle^z g), \quad f, g \in \mathcal{S},$$

analytic in z and bounded in any strip $c_0 \leq \operatorname{Re} z \leq c_1$. As we have seen, this function is bounded by $C||f|| \, ||g||$ if $\operatorname{Re} z = n$ and of course if $\operatorname{Re} z = 0$. In view of

the Hadamard three lines theorem (see, e.g., [5]) this implies that the same bound is true for $z = l$. $\square$

Consider now a more general operator

$$\langle \xi \rangle^l a(x) \langle x \rangle^{-r} b(\xi) \langle \xi \rangle^{-l} \langle x \rangle^r. \tag{2.3}$$

Proposition 2.4. *Suppose that matrix-functions $a(x)$ and $b(\xi)$ have n bounded derivatives and $0 \le l \le n$, $0 \le r \le n$. Then operator (2.3) defined by its sesquilinear form on the Schwartz class $\mathcal{S}$ is bounded.*

Proof. Set $\tilde{a}(x) = a(x)\langle x \rangle^{-r}$. Similarly to the proof of Proposition 2.3, we consider the function

$$(\tilde{a}(x) b(\xi) \langle \xi \rangle^{-z} \langle x \rangle^r f, \langle \xi \rangle^z g), \quad f, g \in \mathcal{S},$$

analytic in z and bounded in any strip $c_0 \le \operatorname{Re} z \le c_1$. In view of the Hadamard three lines theorem it suffices to verify that this function is bounded by $C||f||||g||$ for $\operatorname{Re} z = n$ and for $\operatorname{Re} z = 0$. According to (2.2) the operator $\xi_j^n \tilde{a}(x) b(\xi) \langle \xi \rangle^{-n-i\alpha} \langle x \rangle^r$, $\alpha \in \mathbb{R}$, is a sum of terms

$$(\partial^m \tilde{a}(x)/\partial x_j^m \langle x \rangle^r) \cdot (\langle x \rangle^{-r} (b(\xi) \xi_j^{n-m} \langle \xi \rangle^{-n-i\alpha}) \langle x \rangle^r) \tag{2.4}$$

where $m = 0, 1, \ldots, n$. The first factor here is a bounded function of x. The function $b(\xi)\xi_j^{n-m}\langle \xi \rangle^{-n-i\alpha}$ is bounded, together with its n derivatives, uniformly in α. Therefore, applying Proposition 2.3 with the roles of the variables x and ξ interchanged to the second factor in (2.4), we see that this operator is bounded uniformly in α. Similarly, the operators $\langle x \rangle^{-r} b(\xi) \langle \xi \rangle^{-i\alpha} \langle x \rangle^r$ are also bounded uniformly in α. $\square$

4. let us consider self-adjoint operators H_0 and H in Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}$, respectively. Recall that the essential spectrum σ^{ess} of H is defined as its spectrum σ without isolated eigenvalues of finite multiplicity. The same objects for the operator H_0 will be always labelled by the index '0'. According to the Weyl theorem $\sigma^{ess} = \sigma_0^{ess}$ if $\mathcal{H}_0 = \mathcal{H}$ and the difference $H - H_0$ is a compact operator. We note a simple generalization of this result. Below we use the notation $R_0(z) = (H_0 - z)^{-1}$, $R(z) = (H - z)^{-1}$.

Proposition 2.5. *Let an operator $J : \mathcal{H}_0 \to \mathcal{H}$ be bounded, and let the inverse operator J^{-1} exist and be also bounded. Suppose that*

$$R(z)J - JR_0(z) \in \mathfrak{S}_\infty \tag{2.5}$$

for some point $z \notin \sigma_0 \cup \sigma$. Then $\sigma^{ess} = \sigma_0^{ess}$.

Proof. If $\lambda \in \sigma_0^{ess}$, then there exists a sequence (Weyl sequence) f_n such that $H_0 f_n - \lambda f_n \to 0$, $f_n \to 0$ weakly as $n \to \infty$ and $||f_n|| \ge c > 0$. This ensures that $R_0(z) f_n - (\lambda - z)^{-1} f_n \to 0$ and hence $JR_0(z) f_n - (\lambda - z)^{-1} J f_n \to 0$. Now it follows from condition (2.5) that

$$R(z) J f_n - (\lambda - z)^{-1} J f_n \to 0. \tag{2.6}$$

Set $g_n = R(z) J f_n$. Relation (2.6) means that $H g_n - \lambda g_n \to 0$. Moreover, $g_n \to 0$ weakly as $n \to \infty$ and, by virtue of (2.6), the relation $||g_n|| \to 0$ would have

implied that $||Jf_n|| \to 0$ and therefore $||f_n|| \to 0$. Thus, g_n is a Weyl sequence for the operator H and the point λ so that $\sigma_0^{ess} \subset \sigma^{ess}$.

To prove the opposite inclusion, we remark that

$$J^{-1}R(z) - R_0(z)J^{-1} \in \mathfrak{S}_\infty$$

and use the result already obtained with the roles of H_0, H interchanged and J^{-1} in place of J. $\square$

3. Scattering with two Hilbert spaces

1. Scattering theory requires classification of the spectrum in terms of the theory of measure. Let H be an arbitrary self-adjoint operator in a Hilbert space $\mathcal{H}$. We denote by $E(\cdot)$ its spectral measure. Recall that there is a decomposition $\mathcal{H} = \mathcal{H}^{ac} \oplus \mathcal{H}^{sc} \oplus \mathcal{H}^{pp}$ into the orthogonal sum of invariant subspaces of the operator H such that the measures $(E(\cdot)f, f)$ are absolutely continuous, singular continuous or pure point for all $f \in \mathcal{H}^{ac}$, $f \in \mathcal{H}^{sc}$ or $f \in \mathcal{H}^{pp}$, respectively. The operator H restricted to $\mathcal{H}^{ac}$, $\mathcal{H}^{sc}$ or $\mathcal{H}^{pp}$ shall be denoted H^{ac}, H^{sc} or H^{pp}, respectively. The pure point part corresponds to eigenvalues. The singular continuous part is typically absent. Scattering theory studies the absolutely continuous part H^{ac} of H. We denote P the orthogonal projection onto the absolutely continuous subspace $\mathcal{H}^{ac}$.

Let us consider the large time behaviour of solutions

$$u(t) = e^{-iHt}f.$$

of the time-dependent equation

$$i\frac{\partial u}{\partial t} = Hu, \quad u(0) = f \in \mathcal{H}.$$

If f is an eigenvector, $Hf = \lambda f$, then $u(t) = e^{-i\lambda t}f$, so the time behaviour is evident. By contrast, if $f \in \mathcal{H}^{ac}$, one cannot, in general, calculate $u(t)$ explicitly, but scattering theory allows us to find its asymptotics as $t \to \pm\infty$. In the perturbation theory setting, it is natural to understand the asymptotics of u in terms of solutions of the unperturbed equation, $iu_t = H_0 u$. To compare the operators H_0 and H, one has to introduce an 'identification' operator $J : \mathcal{H}_0 \to \mathcal{H}$ which we suppose to be bounded. Suppose also that, in some sense, J is close to a unitary operator and the perturbation $HJ - JH_0$ is 'small'. Then it turns out that for all $f \in \mathcal{H}^{ac}$, there are $f_0^{\pm} \in \mathcal{H}_0^{ac}$ such that

$$\lim_{t\to\pm\infty} \left\| e^{-iHt}f - Je^{-iH_0 t}f_0^{\pm} \right\| = 0. \tag{3.1}$$

Hence $f_0^{\pm}$ and f are related by the equality

$$f = \lim_{t\to\pm\infty} e^{iHt}Je^{-iH_0 t}f_0^{\pm},$$

which justifies the following fundamental definition. It goes back to C. Møller for $\mathcal{H}_0 = \mathcal{H}$ and $J = I$. In the general case it was formulated by T. Kato [8].

Definition 3.1. Let J be a bounded operator. Then the wave operator $W_\pm(H, H_0, J)$ is defined by

$$W_\pm(H, H_0; J) = \text{s-}\lim_{t\to\pm\infty} e^{iHt} J e^{-iH_0 t} P_0, \tag{3.2}$$

when this limit exists.

Clearly, relation (3.1) holds for all f from the range $\operatorname{Ran} W_\pm$ of the wave operator $W_\pm = W_\pm(H, H_0; J)$. The wave operators enjoy the intertwining property

$$W_\pm(H, H_0; J) H_0 = H W_\pm(H, H_0; J).$$

In our applications they are isometric on $\mathcal{H}_0^{ac}$ which is guaranteered by the condition

$$\text{s-}\lim_{t\to\pm\infty} (J^* J - I) e^{-iH_0 t} P_0 = 0.$$

This implies that H_0^{ac} is unitarily equivalent, via $W_\pm$, to the restriction of H on the range $\operatorname{Ran} W_\pm$ of the wave operator $W_\pm$ and hence $\operatorname{Ran} W_\pm \subset \mathcal{H}^{ac}$.

Definition 3.2. Suppose that the wave operator $W_\pm(H, H_0; J)$ exists and is isometric on $\mathcal{H}_0^{ac}$. It is said to be complete if

$$\operatorname{Ran} W_\pm(H, H_0; J) = \mathcal{H}^{ac}.$$

Thus, if $W_\pm(H, H_0, J)$ exists, is isometric and complete, then H_0^{ac} and H^{ac} are unitarily equivalent.

Suppose additionally that J is boundedly invertible. It is a simple result that $W_\pm(H, H_0, J)$ is complete if and only if the 'inverse' wave operator $W_\pm(H_0, H; J^{-1})$ exists.

2. Let us now discuss conditions for the existence of wave operators. In the abstract framework they are given by the trace-class theory. Its fundamental result is the following theorem of Kato-Rosenblum-Pearson.

Theorem 3.3. *Suppose that H_0 and H are selfadjoint operators in spaces $\mathcal{H}_0$ and $\mathcal{H}$, respectively, $J : \mathcal{H}_0 \to \mathcal{H}$ is a bounded operator and $V = HJ - JH_0 \in \mathfrak{S}_1$. Then the WO $W_\pm(H, H_0; J)$ exist.*

Actually, this result was established by T. Kato and M. Rosenblum for the case $\mathcal{H}_0 = \mathcal{H}$, $J = I$ and then extended by D. Pearson to the operators acting in different spaces.

Applications to differential operators require generalizations of this result. The following result of [1] gives efficient conditions guaranteeing the existence of wave operators and all their properties discussed above. Its simplified proof relying on Theorem 3.3 can be found in [13].

Theorem 3.4. *Suppose that the operator $J : \mathcal{H}_0 \to \mathcal{H}$ has a bounded inverse and $J\mathcal{D}(H_0) = \mathcal{D}(H)$. Suppose that*

$$E(\Lambda)(HJ - JH_0)E_0(\Lambda) \in \mathfrak{S}_1 \tag{3.3}$$

and

$$(J^* J - I)E_0(\Lambda) \in \mathfrak{S}_\infty \tag{3.4}$$

for any bounded interval Λ. *Then the WO* $W_\pm(H, H_0; J)$ *exist, are isometric on* $\mathcal{H}_0^{ac}$*, and are complete. Moreover, there exist the WO* $W_\pm(H_0, H; J^*)$ *and* $W_\pm(H_0, H; J^{-1})$*; these WO are equal to one another and to the operator* $W_\pm^*(H, H_0; J)$*; they are isometric on* $\mathcal{H}^{ac}$ *and complete.*

Thus, the absolutely continuous part of a self-adjoint operator is stable under fairly general perturbations. However assumptions on perturbations are much more restrictive than those required for stability of the essential spectrum (cf. Proposition 2.5).

4. Scattering problems for perturbations of a medium

By definition, a matrix pseudodifferential operator with constant coefficients acts in the momentum representation as multiplication by some matrix-function. This function is called symbol of such pseudodifferential operator.

1. Let $L_2(\mathbb{R}^d; \mathbb{C}^k)$ and let $P(D) = \Phi^* A \Phi$ where A is multiplication by a a symmetric $k \times k$- matrix-function $A(\xi)$. The operator A is selfadjoint on domain $\mathcal{D}(A)$ which consists of functions $\hat{f} \in L_2(\mathbb{R}^d; \mathbb{C}^k)$ such that $A\hat{f} \in L_2(\mathbb{R}^d; \mathbb{C}^k)$. Hence the operator $P(D)$ is selfadjoint on domain $\mathcal{D}(P(D)) = \Phi^* \mathcal{D}(A)$. Below we need to restrict the class of operators $P(D)$. Set

$$\nu(\xi) = \min_{|\mathbf{n}|=1} |A(\xi)\mathbf{n}|, \quad \mathbf{n} \in \mathbb{C}^k.$$

Clearly, $\nu(\xi)$ is the smallest of the absolute values of eigenvalues of the matrix $A(\xi)$. The operator $P(D)$ is called strongly Carleman if $\nu(\xi) \to \infty$ as $|\xi| \to \infty$. We often need a stronger condition

$$\nu(\xi) \geq c|\xi|^{\varkappa}, \quad \varkappa > 0, \quad c > 0, \tag{4.1}$$

for $|\xi|$ sufficiently large.

Clearly, $P(D)$ is a differential operator if entries of the matrix $A(\xi)$ are polynomials of ξ. Let us denote by $A_0(\xi)$ the principal symbol of $A(\xi)$, that is $A_0(\xi)$ consists of terms of higher order which will be denoted by $\operatorname{ord} P(D)$. If $\det A_0(\xi) \neq 0$ for $\xi \neq 0$, then $P(D)$ is called elliptic of order $\operatorname{ord} P(D)$. For elliptic operators condition (4.1) is satisfied with $\varkappa = \operatorname{ord} P(D)$.

Let $M_0(x)$ and $M(x)$ be symmetric $k \times k$ - matrices satisfying the condition

$$0 < c_0 \leq M_0(x) \leq c_1 < \infty, \quad 0 < c_0 \leq M(x) \leq c_1 < \infty, \tag{4.2}$$

and let M_0 and M be the operators of multiplication by these matrices. We denote by $\mathcal{H}$ the Hilbert space with scalar product

$$(f, g)_{\mathcal{H}} = \int_{\mathbb{R}^d} \langle M(x) f(x), g(x) \rangle_{\mathbb{C}^k} \, dx. \tag{4.3}$$

The space $\mathcal{H}_0$ is defined quite similarly with $M(x)$ replaced by $M_0(x)$. Of course the spaces $\mathcal{H}_{00} = L_2(\mathbb{R}^d; \mathbb{C}^k)$, $\mathcal{H}_0$ and $\mathcal{H}$ consist of the same elements. The operators M_0 and M can be considered in all these spaces. The operators H_0 and H are defined by the equalities (1.1) on common domain $\mathcal{D}(H_0) = \mathcal{D}(H) = \mathcal{D}(P(D))$ in the spaces $\mathcal{H}_0$ and $\mathcal{H}$, respectively. Their selfadjointness follows from selfadjointness of the operator $P(D)$ in the space $L_2(\mathbb{R}^d; \mathbb{C}^k)$. Let $I_0 : \mathcal{H}_0 \to \mathcal{H}$ and $I_1 = I_0^{-1}$:

$\mathcal{H} \to \mathcal{H}_0$ be the identical mappings. They are often omitted if this does not lead to any confusion. Note however that

$$I_0^* = M_0^{-1} M, \quad I_1^* = M^{-1} M_0. \tag{4.4}$$

Put $H_{00} = P(D)$, $R_{00}(z) = (H_{00} - z)^{-1}$. Below we use the resolvent identities

$$R(z) = R_{00}(z)(M + z(M - I)R(z)), \quad z \notin \sigma_{00} \cup \sigma, \tag{4.5}$$

$$R(z) = (I - zR(z)(M^{-1} - I))R_{00}(z)M, \quad z \notin \sigma_{00} \cup \sigma, \tag{4.6}$$

which can be verified by a direct calculation. Of course similar identities hold for $R_0(z)$.

As far as the essential spectrum is concerned, we have the following standard assertion.

Proposition 4.1. *Suppose that H_{00} is strongly Carleman and*

$$V(x) := M(x) - M_0(x) \to 0$$

as $|x| \to \infty$. Then $\sigma^{ess} = \sigma_0^{ess}$.

Proof. Let us use the resolvent identity for the pair H_0, H:

$$R(z) - R_0(z) = R(z)M^{-1}V(I + zR_0(z)), \quad z \notin \sigma_0 \cup \sigma. \tag{4.7}$$

According to identity (4.6) and Proposition 2.2, the operators $R(z)M^{-1}V$ and hence (4.7) are compact. Thus it remains to refer to Proposition 2.5. □

2. Let us pass to scattering theory. We proced from the following analytical result.

Proposition 4.2. *Let $P(D)$ be an elliptic differential operator of order $\varkappa$ in the space $\mathcal{H} = L_2(\mathbb{R}^d; \mathbb{C}^k)$. Suppose that the function $M(x)$ obeys condition (4.2). Set $H = M^{-1}P(D)$. Then the operator $\langle x\rangle^{-r} R^n(z)$, $n = 1, 2, \dots$, $z \notin \sigma$, belongs to the class $\mathfrak{S}_p$ provided $p \geq 1$ and*

$$p > d/\min\{r, \varkappa n\} =: p(r, n).$$

Proof. The proof proceeds by induction in n. If $n = 1$, then we use the equality

$$\langle x\rangle^{-r} R = (\langle x\rangle^{-r} \langle \xi\rangle^{-\varkappa}) \cdot (\langle \xi\rangle^{\varkappa} R_{00}) \cdot ((H_{00} - z)R). \tag{4.8}$$

In the right-hand side the first factor belongs to the class $\mathfrak{S}_p$ for $p > p(r, 1)$ according to Proposition 2.2. The second factor is a bounded operator according to (4.1), and the last factor is a bounded operator according to (4.5).

To justify the passage from n to $n+1$, we write the operator $\langle x\rangle^{-r} R^{n+1}$ as

$$\begin{aligned} \langle x\rangle^{-r} R^{n+1} &= (\langle x\rangle^{-r_0} R_{00}) \\ &\times ((H_{00} - z)\langle x\rangle^{-r_1} R_{00} \langle x\rangle^{r_1}) \times (\langle x\rangle^{-r_1}(H_{00} - z)R^{n+1}), \end{aligned} \tag{4.9}$$

where $r_0 = r(n+1)^{-1}$, $r_1 = nr_0$. The first factor here belongs to the class $\mathfrak{S}_p$ for $p > p(r_0, 1)$. The second factor is bounded according to Proposition 2.4. It follows from identity (4.5) that the last factor

$$\langle x\rangle^{-r_1}(H_{00} - z)R^{n+1} = \langle x\rangle^{-r_1} M R^n + z\langle x\rangle^{-r_1}(M - I)R^{n+1}. \tag{4.10}$$

This operator belongs to the class $\mathfrak{S}_p$ where, by the inductive assumption, $p > p(r_1, n)$. Thus, by Proposition 2.1, the product (4.9) belongs to the class $\mathfrak{S}_p$ where

$$p^{-1} < p(r_0, 1)^{-1} + p(r_1, n)^{-1} = (n+1)p(r_0, 1)^{-1} = p(r, n+1)^{-1}$$

and of course $p \geq 1$. $\square$

Now it is easy to prove

Theorem 4.3. *Let $P(D)$ be an elliptic differential operator of order $\varkappa$ in the space $\mathcal{H} = L_2(\mathbb{R}^d; \mathbb{C}^k)$. Assume that $M_0(x)$ and $M(x)$ satisfy conditions* (4.2) *and* (1.2). *Then the wave operators*

$$W_\pm(H, H_0; I_0), \quad W_\pm(H_0, H; I_0^*) \quad \text{and} \quad W_\pm(H_0, H; I_1)$$

exist, are isometric and are complete.

Proof. By Theorem 3.4, we have only to check inclusions (3.3) and (3.4), that is

$$E(\Lambda)(HI_0 - I_0H_0)E_0(\Lambda) \in \mathfrak{S}_1 \tag{4.11}$$

and

$$(I_0^* I_0 - I)E_0(\Lambda) \in \mathfrak{S}_\infty \tag{4.12}$$

for an arbitrary bounded interval Λ. It follows from (1.1) and (4.4) that

$$HI_0 - I_0H_0 = -M^{-1}VH_0, \tag{4.13}$$

and

$$I_0^* I_0 - I = M_0^{-1}V. \tag{4.14}$$

By Proposition 4.2, $VR_0^n \in \mathfrak{S}_1$ if $n\varkappa > d$. Since the operators $H_0^n E_0(\Lambda)$ are bounded for all n, this implies both inclusions (4.11) and (4.12) (actually, the operator in (4.12) also belongs to the trace class). $\square$

Remark 4.4. Actually, we have verified that $(HI_0 - I_0H_0)E_0(\Lambda) \in \mathfrak{S}_1$ which is stronger than (4.11). Inclusion (4.11) follows also from the inclusions $\langle x\rangle^{-r}E_0(\Lambda) \in \mathfrak{S}_2$ and $\langle x\rangle^{-r}E(\Lambda) \in \mathfrak{S}_2$ for $r > d/2$.

5. Wave equation

A propagation of sound waves in inhomogeneous media is often described by the wave equation. Basically, the methods of the previous section are applicable to this case. However, by a natural reduction of the wave equation to the Schrödinger equation, the pseudodifferential operators with non-smooth symbols appear. This requires a modification of Theorem 4.3. Here we use the same notation as in the previous section.

1. Let us consider the equation

$$m(x)\frac{\partial^2 u(x,t)}{\partial t^2} = \Delta u(x,t), \quad x \in \mathbb{R}^d, \tag{5.1}$$

where the function $m(x)$ satisfies condition (4.2). Set

$$\mathbf{u}(x,t) = \begin{pmatrix} ((-\Delta)^{1/2}u)(x,t) \\ \partial u(x,t)/\partial t \end{pmatrix}, \quad M(x) = \begin{pmatrix} I & 0 \\ 0 & m(x) \end{pmatrix}. \tag{5.2}$$

Then equation (5.1) is equivalent to the equation

$$iM(x)\frac{\partial \mathbf{u}(x,t)}{\partial t} = (-\Delta)^{1/2} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \mathbf{u}(x,t). \tag{5.3}$$

According to (5.2) initial data for equations (5.1) and (5.3) are connected by the relation $\mathbf{u}(0) = (((-\Delta)^{1/2}u)(0), u_t(0))^t$ (the index 't' means 'transposed').

Set

$$P(D) = (-\Delta)^{1/2} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},$$

and denote by $\mathcal{H}$ the Hilbert space with scalar product (4.3) where $k = 2$. The operator $H = M^{-1}P(D)$ is selfadjoint in the space $\mathcal{H}$. Unitarity of the operator $\exp(-iHt)$ in this space is equivalent to the conservation of the energy

$$||(-\Delta)^{1/2}u(t)||^2 + (mu_t(t), u_t(t)).$$

Suppose now that another function $m_0(x)$ also satisfying condition (4.2) is given. All objects constructed by this function will be labelled by '0'. Let $u_0(x,t)$ be a solution of equation (5.1) with $m(x)$ replaced by $m_0(x)$. Our goal is to compare the asymptotics for large t of solutions $u(x,t)$ and $u_0(x,t)$ in the energy norm. This can be done in terms of the wave operators for the pair H_0, H. Indeed, we have the following obvious result.

Proposition 5.1. *Let $f = (((-\Delta)^{1/2}u)(0), u_t(0))^t$, $f_0 = (((-\Delta)^{1/2}u_0)(0), u_{0,t}(0))^t$ and let $t \to \infty$ (or $t \to -\infty$). Then relations*

$$|| \exp(-iHt)f - I_0 \exp(-iH_0t)f_0||_{\mathcal{H}} \to 0$$

and

$$||(-\Delta)^{1/2}(u(t) - u_0(t))|| \to 0, \quad ||u_t(t) - u_{0,t}(t))|| \to 0$$

are equivalent to each other.

2. According to Proposition 5.1 scattering theory for the wave equation reduces to a proof of the existence and completeness of the wave operators $W_\pm(H, H_0; I_0)$. Now the symbol of the operator $P(D)$ equals

$$A(\xi) = |\xi| \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

This function has a singularity at $\xi = 0$ so that Theorem 4.3 cannot be directly applied. Since however this singularity is not too strong, we have the following result.

Theorem 5.2. *Let $d \leq 3$. Assume that functions $m_0(x)$ and $m(x)$ satisfy the condition*

$$0 < c_0 \leq m_0(x) \leq c_1 < \infty, \quad 0 < c_0 \leq m(x) \leq c_1 < \infty$$

and that

$$|m(x) - m_0(x)| \leq C(1+|x|)^{-\rho}, \quad \rho > d.$$

Then all conclusions of Theorem 4.3 *hold.*

Proof. Again by Theorem 3.4, we have only to verify the inclusions (4.11) and (4.12). It follows from (4.13) and (4.14) that it suffices to verify the inclusion

$$\langle x \rangle^{-r} E(\Lambda) \in \mathfrak{S}_2, \quad r = \rho/2 > d/2, \tag{5.4}$$

and the same inclusion for the operator H_0. The operators H_0 and H are quite symmetric, and hence we have to check (5.4) only.

Let first $d = 1$. The operator $\langle x \rangle^{-r} \langle \xi \rangle^{-1} \in \mathfrak{S}_2$ so that, by virtue of (4.8) where $\varkappa = 1$, the operator $\langle x \rangle^{-r} R$ is also Hilbert-Schmidt.

In the cases $d = 2$ and $d = 3$ we check that

$$\langle x \rangle^{-r} R^2 \in \mathfrak{S}_2. \tag{5.5}$$

Using again (4.9), (4.10) for $n = 1$ and $r_0 = r_1 = r/2$, we see that

$$\begin{aligned} \langle x \rangle^{-r} R^2 \quad &= \quad (\langle x \rangle^{-r/2} R_{00}) \, ((H_{00} - z) \langle x \rangle^{-r/2} R_{00} \langle x \rangle^{r/2}) \\ &\times \quad (\langle x \rangle^{-r/2} (M + z(M - I)R)R). \end{aligned}$$

By Proposition 2.2, the operators $\langle x \rangle^{-r/2} R_{00}$ and hence $\langle x \rangle^{-r/2} R$ belong to the class $\mathfrak{S}_4$. Therefore it remains to notice that the function $(A(\xi) - z)^{-1} \langle \xi \rangle$ is bounded together with its first derivatives so that, by Proposition 2.4,

$$(H_{00} - z) \langle x \rangle^{-r/2} R_{00} \langle x \rangle^{r/2} \in \mathfrak{B}, \quad r \leq 2.$$

This yields (5.5). □

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